UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. M.Sci.

Mathematics M11B: Analysis 2

COURSE CODE	:	**MATHM11B**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-00**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

00-C0985-3-140

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let f and g be functions defined on $\mathbb{R}$ which are differentiable at a. Show that

   (i) $(f+g)'(a) = f'(a) + g'(a)$;
   (ii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$;
   (iii) if $g'(a) \neq 0$, $\quad \left(\dfrac{1}{g}\right)'(a) = \dfrac{-g'(a)}{(g(a))^2}$.

   Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

   Show that $f'(x)$ exists for all x but f' is not continuous at 0.

2. (a) Define $\left(\begin{array}{c} \alpha \\ n \end{array}\right) = \dfrac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$, α real, n a positive integer. Show that

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array}\right) x^k, \quad \text{for } |x| < 1.$$

   (b) If f is a function on $\mathbb{R}$ and

$$|f(x) - f(y)| \leq (x-y)^2 \text{ for all real } x, y$$

   show that f is constant.

3. (a) Let f be a continuous and strictly increasing function on $[a, b]$. Show that f has an inverse f^{-1} which is continuous and strictly increasing on $[f(a), f(b)]$. Show further that if f is differentiable at some point c, $a < c < b$, with $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$ with $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$.

   (b) Show that $f(x) = \tan x$, $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ has an inverse on $(-\infty, \infty)$ and find the derivative of the inverse.

4. (a) Let $\{[a_n, b_n]\}_{n=1}^{\infty}$ be a nested sequence of non-empty closed intervals i.e. $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$, $n = 1, 2, \ldots$. Show that $\cap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$. Is the same result true for open intervals?

(b) Let $\{I_\alpha\}_{\alpha \in A}$ be a collection of open intervals whose union covers the closed interval $[a, b]$. Show that there exists a finite sub-collection of $\{I_\alpha\}_{\alpha \in A}$ whose union covers $[a, b]$. Is the same result true for an open interval (a, b)?

5. If a function f is continuous on $[a, b]$, show that

(a) f is uniformly continuous on $[a, b]$;

(b) f is Riemann integrable on $[a, b]$.

6. (a) Let f be a Riemann integrable function on $[a, b]$ and g a continuous function on $\mathbb{R}$. Show that $h(x) = g(f(x))$ is Riemann integrable on $[a, b]$.

(b) If f is Riemann integrable on $[a, b]$ and $\dfrac{1}{|f|}$ is bounded on $[a, b]$, show that $\dfrac{1}{|f|}$ is Riemann integrable on $[a, b]$.

7. (a) Let f be a continuous function on $[1, \infty)$ with

(i) $f(x) \geq 0, \quad x \geq 1$;

(ii) $f(x)$ decreasing, $x \geq 1$;

(iii) $f(x) \to 0$ as $x \to \infty$.

Then, if $T_n = \sum_{r=1}^{n} f(r) - \int_1^n f(x)dx$, show that

$$f(n) \leq T_n \leq f(1)$$

and that $[T_n]_{n=1}^{\infty}$ is a decreasing sequence converging to T say, where $0 \leq T \leq f(1)$.

(b) Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges.